74. Given an $\epsilon > 0$, by definition of convergence there corresponds an N such that for all $n < N, |L_1 - a_n| < \epsilon$ and $|L_2 - a_n| < \epsilon$. (There is one such number for each series, and we may let N be the larger of the two numbers.) Now

$$\begin{aligned} |L_2 - L_1| &= |L_2 - a_n + a_n - L_1| \\ &\le |L_2 - a_n| + |a_n - L_n| \\ &< \epsilon + \epsilon \\ &= 2\epsilon \end{aligned}$$

$|L_2 - L_1| < 2\epsilon$ says that the difference between two fixed values is smaller than any positive number 2ϵ. The only nonnegative number smaller than every positive number is 0, so $|L_2 - L_1| = 0$ or $L_1 = L_2$.

75. Consider the two subsequences $a_{k(n)}$ and $a_{i(n)}$, where $\lim_{n\to\infty} a_{k(n)} = L_1$, $\lim_{n\to\infty} a_{i(n)} = L_2$, and $L_1 \ne L_2$. Given an $\epsilon > 0$ there corresponds an N_1 such that for $k(n) > N_1$, $|a_{k(n)} - L_1| < \epsilon$, and an N_2 such that for $i(n) > N_2$, $|a_{i(n)} - L_2| < \epsilon$. Assume a_n converges. Let $N = \max\{N_1, N_2\}$. Then for $n > N$, we have that $|a_n - L_1| < \epsilon$ and $|a_n - L_2| < \epsilon$ for infinitely many n. This implies that $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} a_n = L_2$ where $L_1 \ne L_2$. Since the limit of a sequence is unique (by Exercise 55), a_n does not converge and hence diverges.

76. (a) $\lim_{n\to\infty} \dfrac{3n+1}{n+1} = 3$

(b) The line y = 3 is a horizontal asymptote of the graph of the function $f(x) = \dfrac{3x+1}{x+1}$, which means $\lim_{x\to\infty} f(x) = 3$. Because $f(n) = a_n$ for all positive integers n, it follows that $\lim_{n\to\infty} a_n$ must also be 3.

Section 9.2 Taylor Series (pp. 484–494)

Exploration 1 Designing a Polynomial to Specifications

1. Since $P(0) = 1$, we know that the constant coefficient is 1. Since $P'(0) = 2$, we know that the coefficient of x is 2.

Since $P''(0) = 3$, we know that the coefficient of x^2 is $\frac{3}{2}$. (The 2 in the denominator is needed to cancel the factor of 2 that results from differentiating x^2.) Similarly, we find the coefficients of x^3 and x^4 to be $\frac{4}{6}$ and $\frac{5}{24}$.

Thus, $P(x) = 1 + 2x + \frac{3}{2}x^2 + \frac{4}{6}x^3 + \frac{5}{24}x^4$.

Exploration 2 A Power Series for the Cosine

1. $\cos(0) = 1$

$\cos'(0) = -\sin(0) = 0$

$\cos''(0) = -\cos(0) = -1$

$\cos^{(3)}(0) = \sin(0) = 0$

etc.

The pattern 1, 0, –1, 0 will repeat forever. Therefore,

$P_6(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}$, and the Taylor series is

$$1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots.$$

2. A clever shortcut is simply to differentiate the previously-discovered series for sin x term-by-term!

Exploration 3 Approximating sin 13

1. 0.4201670368...

4. 20 terms.

Quick Review 9.2

1. $f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

$f''(x) = 4e^{2x}$

$f'''(x) = 8e^{2x}$

$f^{(n)}(x) = 2^n e^{2x}$

2. $f(x) = \dfrac{1}{x-1}$

$f'(x) = -(x-1)^{-2}$

$f''(x) = 2(x-1)^{-3}$

$f'''(x) = -6(x-1)^{-4}$

$f^{(n)}(x) = (-1)^n n!(x-1)^{-(n+1)}$

3. $f(x) = 3^x$

$f'(x) = 3^x \ln 3$

$f''(x) = 3^x (\ln 3)^2$

$f'''(x) = 3^x (\ln 3)^3$

$f^{(n)}(x) = 3^x (\ln 3)^n$

4. $f(x) = \ln(x)$

$f'(x) = x^{-1}$

$f''(x) = -x^{-2}$

$f'''(x) = 2x^{-3}$

$f^{(4)}(x) = -6x^{-4}$

$f^{(n)}(x) = (-1)^{n-1}(n-1)!x^{-n}$ for $n \ge 1$

5. $f(x)=x^n$

$f'(x)=nx^{n-1}$

$f''(x)=n(n-1)x^{n-2}$

$f'''(x)=n(n-1)(n-2)x^{n-3}$

$f^{(k)}(x)=\frac{n!}{(n-k)!}x^{n-k}$

$f^{(n)}(x)=\frac{n!}{0!}x^0=n!$

6. $\frac{dy}{dx}=\frac{d}{dx}\frac{x^n}{n!}=\frac{nx^{n-1}}{n!}=\frac{x^{n-1}}{(n-1)!}$

7. $\frac{dy}{dx}=\frac{d}{dx}\frac{2^n(x-a)^n}{n!}=\frac{2^n n(x-a)^{n-1}}{n!}=\frac{2^n(x-a)^{n-1}}{(n-1)!}$

8. $\frac{dy}{dx}=\frac{d}{dx}\left[(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right]=(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)!}=\frac{(-1)^n x^{2n}}{(2n)!}$

9. $\frac{dy}{dx}=\frac{d}{dx}\frac{(x+a)^{2n}}{(2n)!}=\frac{2n(x+a)^{2n-1}}{(2n)!}=\frac{(x+a)^{2n-1}}{(2n-1)!}$

10. $\frac{dy}{dx}=\frac{d}{dx}\frac{(1-x)^n}{n!}=\frac{n(1-x)^{n-1}(-1)}{n!}=-\frac{(1-x)n-1}{(n-1)!}$

Section 9.2 Exercises

1. $P(0)=\sqrt{1+x^2}=1$

$P'(0)=\frac{x}{\sqrt{x^2+1}}=0$

$P''(0)=\frac{1}{(x^2+1)^{3/2}}=1$

$P'''(0)=\frac{-3x}{(x^2+1)^{5/2}}=0$

$P^{(4)}(0)=\frac{3(4x^2-1)}{(x^2+1)^{7/2}}=-3$

$P_4(x)=-\frac{1}{8}x^4+\frac{1}{2}x^2+1$

2. $P(0)=e^{2x}=1$

$P'(0)=2e^{2x}=2$

$P''(0)=4e^{2x}=4$

$P'''(0)=8e^{2x}=8$

$P^{(4)}(0)=16e^{2x}=16$

$P_4(x)=\frac{2}{3}x^4+\frac{4}{3}x^3+2x^2+2x+1$

3. $P(0)=\frac{1}{x+2}=\frac{1}{2}$

$P'(0)=-\frac{1}{(x+2)^2}=-\frac{1}{4}$

$P''(0)=\frac{2}{(x+2)^3}=\frac{1}{4}$

$P'''(0)=-\frac{6}{(x+2)^4}=-\frac{3}{8}$

$P^{(4)}(0)=\frac{24}{(x+2)^5}=\frac{3}{4}$

$P^{(5)}(0)=-\frac{120}{(x+2)^6}=-\frac{15}{8}$

$P_5(x)=-\frac{x^5}{2^6}+\frac{x^4}{2^5}-\frac{x^3}{2^4}+\frac{x^2}{2^3}-\frac{x}{2^2}+\frac{1}{2},\ \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{2^{n+1}}$

4. $P(0)=e^1=e$

$P'(0)=-e^{1-x}=-e$

$P''(0)=e^{1-x}=e$

$P'''(0)=e^{1-x}=-e$

$P^{(4)}(0)=e^{1-x}=e$

$P^{(5)}(0)=-e^{1-x}=-e$

$P_5(x)=-e\frac{x^5}{5!}+e\frac{x^4}{4!}-e\frac{x^3}{3!}+e\frac{x^2}{2!}-e\,x+e,\ \sum_{n=0}^{\infty}(-1)^n e\frac{x^n}{n!}$

5. Substitute $2x$ for x in the Maclaurin series for $\sin x$ shown at the end of Section 9.2.

$$\sin 2x=2x-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}-\cdots+(-1)^n\frac{(2x)^{2n+1}}{(2n+1)!}+\cdots$$
$$=2x-\frac{4x^3}{3}+\frac{4x^5}{15}-\cdots+\frac{(-1)^n(2x)^{2n+1}}{(2n+1)!}+\cdots$$

This series converges for all real x.

6. Substitute $-x$ for x in the Maclaurin series for $\ln(1+x)$ shown at the end of Section 9.2.

$$\ln(1-x)=(-x)-\frac{(-x)^2}{2}+\frac{(-x)^3}{3}-\cdots+(-1)^{n-1}\frac{(-x)^n}{n}+\cdots$$
$$=-x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots-\frac{x^n}{n}-\cdots$$

This series converges when $-1\le -x<1$, so the interval of convergence is $[-1, 1)$.

7. Substitute x^2 for x in the Maclaurin series for $\tan^{-1}x$ shown at the end of Section 9.2.

$$\tan^{-1}x^2=x^2-\frac{(x^2)^3}{3}+\frac{(x^2)^5}{5}-\cdots+(-1)^n\frac{(x^2)^{2n+1}}{2n+1}+\cdots$$
$$=x^2-\frac{x^6}{3}+\frac{x^{10}}{5}-\cdots+\frac{(-1)^n x^{4n+2}}{2n+1}+\cdots$$

This series converges when $\left|x^2\right|\le 1$, so the interval of convergence is $[1, 1]$.

8. $7xe^x = 7x\left(1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots\right)$

$= 7x+7x^2+\frac{7x^3}{2!}+\cdots+\frac{7x^{n+1}}{n!}\cdot$

This series converges for all real x.

9. $\cos(x+2) = (\cos 2)(\cos x) - (\sin 2)(\sin x)$

$$= (\cos 2)\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots\right]$$

$$-(\sin 2)\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots\right]$$

$$= (\cos 2)-(\sin 2)x-\frac{(\cos 2)x^2}{2!}+\frac{(\sin 2)x^3}{3!}+\frac{(\cos 2)x^4}{4!}$$

$$-\frac{(\sin 2)x^5}{5!}-\cdots$$

We need to write an expression for the coefficient of x^k.

If k is even, the coefficient is $\frac{(-1)^n(\cos 2)}{(2n)!}$ where $2n = k$.

Thus the coefficient is $\frac{(-1)^{k/2}(\cos 2)}{k!}$, which is the same as

$\frac{(-1)^{\text{int}[(k+1)/2]}(\cos 2)}{k!}$. If k is odd, the coefficient is

$\frac{(-1)^{n+1}(\sin 2)}{(2n+1)!}$ 1 where $2n+1 = k$. Thus the coefficient is

$\frac{(-1)^{(k+1)/2}(\sin 2)}{(2n+1)!}$, which is the same as

$\frac{(-1)^{\text{int}[(k+1)/2]}(\cos 2)}{k!}$. Hence the general term is

$\frac{(-1)^A Bx^n}{n!}$, where $A = \text{int}\left(\frac{n+1}{2}\right)$, and $B = \sin 2$ if n is odd and $B = \cos 2$ if n is even.

Another way to handle the general term is to observe that $-\sin 2 = \cos\left(2+\frac{\pi}{2}\right)$, $-\cos 2 = \cos(2+\pi)$, and so on, so the general term is $\left[\frac{1}{n!}\cos\left(2+\frac{n\pi}{2}\right)\right]x^n$. The series converges for all real x.

10. $x^2\cos x = x^2\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots\right)$

$= x^2-\frac{x^4}{2}+\frac{x^6}{24}-\cdots+\frac{(-1)^n x^{2n+2}}{(2n)!}+\cdots$

The series converges for all real x.

11. Factor out x and substitute x^3 for x in the Maclaurin series for $\frac{1}{1-x}$ shown at the end of Section 9.2.

$$\frac{x}{1-x^3} = x\left(\frac{1}{1-x^3}\right)$$

$$= x[1+x^3+(x^3)^2+\cdots+(x^3)^n+\cdots]$$

$$= x+x^4+x^7+\cdots+x^{3n+1}+\cdots$$

The series converges for $|x^3|<1$, so the interval of convergence is $(-1, 1)$.

12. Substitute $-2x$ for x in the Maclaurin series for e^x shown at the end of Section 9.2.

$$e^{-2x} = 1+(-2x)+\frac{(-2x)^2}{2!}+\cdots+\frac{(-2x)^n}{n!}+\cdots$$

$$= 1-2x+2x^2-\cdots+\frac{(-1)^n 2^n x^n}{n!}+\cdots$$

The series converges for all real x.

13. $P(0) = \frac{1}{1-(x-2)} = \frac{1}{3}$

$$P(x) = \sum_{n=0}^{\infty}(-1)^n\frac{(x-2)^n}{3^{n+1}}$$

$$= \frac{1}{3}-\frac{x-2}{3^2}+\frac{(x-2)^2}{3^3}-\frac{(x-2)^3}{3^4}+\cdots$$

14. $P(0) = \frac{1}{1-e^{x/2}} = 1$

$$P(x) = \sum_{n=0}^{\infty}e^{1/2}\frac{(x-1)^n}{2^n n!}$$

$$= e^{1/2}+e^{1/2}\frac{(x-1)}{2}+e^{1/2}\frac{(x-1)^2}{2^2 2!}+e^{1/2}\frac{(x-1)^3}{2^3 3!}+e^{1/2}\frac{(x-1)^4}{2^4 4!}$$

15. (a) Since f is a cubic polynomial, it is its own Taylor polynomial of order 3.

$P_3(x) = x^3-2x+4$ or $4-2x+x^3$

(b) $f(1) = x^3-2x+4\Big|_{x=1} = 3$

$f'(1) = 3x^2-2\Big|_{x=1} = 1$

$f''(1) = 6x\Big|_{x=1} = 6$, so $\frac{f''(1)}{2!} = 3$

$f'''(1) = 6\Big|_{x=1} = 6$, so $\frac{f'''(1)}{3!} = 1$

$P_3(x) = 3+(x-1)+3(x-1)^2+(x-1)^3$

16. (a) Since f is a cubic polynomial, it is its own Taylor polynomial of order 3.

$P_3(x) = 2x^3+x^2+3x-8$ or $-8+3x+x^2+2x^3$

(b) $f(1) = 2x^3+x^2+3x-8\Big|_{x=1} = -2$

$f'(1) = 6x^2+2x+3\Big|_{x=1} = 11$

$f''(1) = 12x+2\Big|_{x=1} = 14$, so $\frac{f''(1)}{2!} = 7$

$f'''(1) = 12\Big|_{x=1} = 12$, so $\frac{f'(1)}{3!} = 2$

$P_3(x) = -2+11(x-1)+7(x-1)^2+2(x-1)^3$

17. (a) Since $f(0)=f'(0)=f''(0)=f'''(0)=0$, the Taylor polynomial of order 3 is $P_3(0)=0$.

(b) $f(1)=x^4\big|_{x=1}=1$

$f'(1)=4x^3\big|_{x=1}=4$

$f''(1)=12x^2\big|_{x=1}=12$, so $\frac{f''(1)}{2!}=6$

$f'''(1)=24x\big|_{x=1}=24$, so $\frac{f'''(1)}{3!}=4$

$P_3(x)=1+4(x-1)+6(x-1)^2+4(x-1)^3$

18. $f(2)=\frac{1}{x}\Big|_{x=2}=\frac{1}{2}$

$f'(2)=-x^{-2}\big|_{x=2}=-\frac{1}{4}$

$f''(2)=2x^{-3}\big|_{x=2}=\frac{1}{4}$, so $\frac{f''(2)}{2!}=\frac{1}{8}$

$f'''(2)=-6x^{-4}\big|_{x=2}=-\frac{3}{8}$, so $\frac{f'''(2)}{3!}=-\frac{1}{16}$

$P_0(x)=\frac{1}{2}$

$P_1(x)=\frac{1}{2}-\frac{x-2}{4}$

$P_2(x)=\frac{1}{2}-\frac{x-2}{4}+\frac{(x-2)^2}{8}$

$P_3(x)=\frac{1}{2}-\frac{x-2}{4}+\frac{(x-2)^2}{8}-\frac{(x-2)^3}{16}$

19. $f\left(\frac{\pi}{4}\right)=\sin x\big|_{x=\pi/4}=\frac{\sqrt{2}}{2}$

$f'\left(\frac{\pi}{4}\right)=\cos x\big|_{x=\pi/4}=\frac{\sqrt{2}}{2}$

$f''\left(\frac{\pi}{4}\right)=-\sin x\big|_{x=\pi/4}=-\frac{\sqrt{2}}{2}$, so $\frac{f''\left(\frac{\pi}{4}\right)}{2!}=-\frac{\sqrt{2}}{4}$

$f'''\left(\frac{\pi}{4}\right)=-\cos x\big|_{x=\pi/4}=-\frac{\sqrt{2}}{2}$, so $\frac{f'''\left(\frac{\pi}{4}\right)}{3!}=-\frac{\sqrt{2}}{12}$

$P_0(x)=\frac{\sqrt{2}}{2}$

$P_1(x)=\frac{\sqrt{2}}{2}+\left(\frac{\sqrt{2}}{2}\right)\left(x-\frac{\pi}{4}\right)$

$P_2(x)=\frac{\sqrt{2}}{2}+\left(\frac{\sqrt{2}}{2}\right)\left(x-\frac{\pi}{4}\right)-\left(\frac{\sqrt{2}}{4}\right)\left(x-\frac{\pi}{4}\right)^2$

$P_3(x)=\frac{\sqrt{2}}{2}+\left(\frac{\sqrt{2}}{2}\right)\left(x-\frac{\pi}{4}\right)-\left(\frac{\sqrt{2}}{4}\right)\left(x-\frac{\pi}{4}\right)^2$
$-\left(\frac{\sqrt{2}}{12}\right)\left(x-\frac{\pi}{4}\right)^3$

20. $f\left(\frac{\pi}{4}\right)=\cos x\big|_{x=\pi/4}=\frac{\sqrt{2}}{2}$

$f'\left(\frac{\pi}{4}\right)=-\sin x\big|_{x=\pi/4}=-\frac{\sqrt{2}}{2}$

$f''\left(\frac{\pi}{4}\right)=-\cos x\big|_{x=\pi/4}=-\frac{\sqrt{2}}{2}$, so $\frac{f''\left(\frac{\pi}{4}\right)}{2!}=-\frac{\sqrt{2}}{4}$

$f'''\left(\frac{\pi}{4}\right)=\sin x\big|_{x=\pi/4}=\frac{\sqrt{2}}{2}$, so $\frac{f'''\left(\frac{\pi}{4}\right)}{3!}=\frac{\sqrt{2}}{12}$

$P_0(x)=\frac{\sqrt{2}}{2}$

$P_1(x)=\frac{\sqrt{2}}{2}-\left(\frac{\sqrt{2}}{2}\right)\left(x-\frac{\pi}{4}\right)$

$P_2(x)=\frac{\sqrt{2}}{2}-\left(\frac{\sqrt{2}}{2}\right)\left(x-\frac{\pi}{4}\right)-\left(\frac{\sqrt{2}}{4}\right)\left(x-\frac{\pi}{4}\right)^2$

$P_3(x)=\frac{\sqrt{2}}{2}-\left(\frac{\sqrt{2}}{2}\right)\left(x-\frac{\pi}{4}\right)-\left(\frac{\sqrt{2}}{4}\right)\left(x-\frac{\pi}{4}\right)^2$
$+\left(\frac{\sqrt{2}}{12}\right)\left(x-\frac{\pi}{4}\right)^3$

21. $f(4)=x^{1/2}\big|_{x=4}=2$

$f'(4)=\frac{1}{2}x^{-1/2}\Big|_{x=4}=\frac{1}{4}$

$f''(4)=-\frac{1}{4}x^{-3/2}\Big|_{x=4}=-\frac{1}{32}$, so $\frac{f''(4)}{2!}=-\frac{1}{64}$

$f'''(4)=\frac{3}{8}x^{-5/2}\Big|_{x=4}=\frac{3}{256}$, so $\frac{f'''(4)}{3!}=\frac{1}{512}$

$P_0(x)=2$

$P_1(x)=2+\frac{x-4}{4}$

$P_2(x)=2+\frac{x-4}{4}-\frac{(x-4)^2}{64}$

$P_3(x)=2+\frac{x-4}{4}-\frac{(x-4)^2}{64}+\frac{(x-4)^3}{512}$

22. (a) $P_3(x)=4+5x+\frac{-8}{2!}x^2+\frac{6}{3!}x^3$

$=4+5x-4x^2+x^3$

$f(0.2)\approx P_3(0.2)=4.848$

(b) Since the Taylor series of $f'(x)$ can be obtained by differentiating the terms of the Taylor series of $f(x)$, the second order Taylor polynomial of $f'(x)$ is given by $5-8x+3x^2$. Evaluating at $x=0.2$, $f'(0.2)\approx 3.52$

23. (a) $P_3(x)=4+(-1)(x-1)+\frac{3}{2!}(x-1)^2+\frac{2}{3!}(x-1)^3$

$=4-(x-1)+\frac{3}{2}(x-1)^2+\frac{1}{3}(x-1)^3$

$f(1.2)\approx P_3(1.2)\approx 3.863$

23. Continued

(b) Since the Taylor series of $f'(x)$ can be obtained by differentiating the terms of the Taylor series of $f(x)$, the second order Taylor polynomial of $f'(x)$ is given by $-1+3(x-1)+(x-1)^2$. Evaluating at $x=1.2$, $f'(1.2)\approx -0.36$

24. (a) Since $f'(0)x=\frac{x}{2!}$, $f'(0)=\frac{1}{2!}=\frac{1}{2}$.

Since $\frac{f^{(10)}(0)}{10!}x^{10}=\frac{x^{10}}{11!}$, $f^{(10)}(0)=\frac{10!}{11!}=\frac{1}{11}$.

(b) Multiply each term of $f(x)$ by x.

$$g(x)=x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots+\frac{x^{n+1}}{(n+1)!}+\cdots$$

(c) $g(x)=e^x-1$

25. (a) Substitute $\frac{x}{2}$ for x in the Maclaurin series for e^x shown at the end of Section 9.2

$$\begin{aligned}e^{x/2}&=1+\frac{x}{2}+\frac{\left(\frac{x}{2}\right)^2}{2}+\cdots+\frac{\left(\frac{x}{2}\right)^n}{n!}+\cdots\\&=1+\frac{x}{2}+\frac{x^2}{8}+\cdots+\frac{x^n}{2^n\cdot n!}\end{aligned}$$

(b)
$$\begin{aligned}g(x)&=\frac{e^x-1}{x}\\&=\frac{1}{x}\left[\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots\right)-1\right]\\&=\frac{1}{x}\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots\right)\\&=1+\frac{x}{2!}+\frac{x^2}{3!}+\cdots+\frac{x^{n-1}}{n!}+\cdots\end{aligned}$$

This can also be written as

$$1+\frac{x}{2!}+\frac{x^2}{3!}+\cdots+\frac{x^n}{(n+1)!}+\cdots.$$

(c)
$$\begin{aligned}g'(x)&=\frac{d}{dx}\frac{e^x-1}{x}=\frac{(x)(e^x)-(e^x-1)(1)}{x^2}\\&=\frac{xe^x-e^x+1}{x^2}\end{aligned}$$

$$g'(1)=\frac{e-e+1}{1}=1$$

From the series,

$$\begin{aligned}g'(x)&=\frac{d}{dx}\left(1+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\cdots+\frac{x^n}{(n+1)!}+\cdots\right)\\&=\frac{1}{2!}+\frac{2x}{3!}+\frac{3x^2}{4!}+\cdots+\frac{nx^{n-1}}{(n+1)!}+\cdots\\&=\sum_{n=1}^{\infty}\frac{nx^{n-1}}{(n+1)!}\end{aligned}$$

Therefore, $g'(1)=\sum_{n=1}^{\infty}\frac{n}{(n+1)!}$, which means

$$\sum_{n=1}^{\infty}\frac{n}{(n+1)!}=1.$$

26. (a) Factor out 2 and substitute t^2 for x in the Maclaurin series for $\frac{1}{1-x}$ at the end of Section 9.2.

$$\begin{aligned}f(t)&=\frac{2}{1-t^2}\\&=2\left(\frac{1}{1-t^2}\right)\\&=2\left[1+t^2+(t^2)^2+(t^2)^3+\cdots+(t^2)^n+\cdots\right]\\&=2+2t^2+2t^4+2t^6+\cdots+2t^{2n}+\cdots\end{aligned}$$

(b) Since $G(0)=0$, the constant term is zero and we may find $G(x)$ by integrating the terms of the series for $f(x)$.

$$G(x)=2x+\frac{2x^3}{3}+\frac{2x^5}{5}+\frac{2x^7}{7}+\cdots+\frac{2x^{2n+1}}{2n+1}+\cdots$$

27. (a)
$$f(0)=(1+x)^{1/2}\Big|_{x=0}=1$$

$$f'(0)=\frac{1}{2}(1+x)^{-1/2}\Big|_{x=0}=\frac{1}{2}$$

$$f''(0)=-\frac{1}{4}(1+x)^{-3/2}\Big|_{x=0}=-\frac{1}{4},\ \text{so}\ \frac{f''(0)}{2!}=-\frac{1}{8}$$

$$f'''(0)=\frac{3}{8}(1+x)^{-5/2}\Big|_{x=0}=\frac{3}{8},\ \text{so}\ \frac{f'''(0)}{3!}=\frac{1}{16}$$

$$P_4(x)=1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}$$

(b) Since $g(x)=f(x)^2$, the first four teams are

$$1+\frac{x^2}{2}-\frac{x^4}{8}+\frac{x^6}{16}.$$

(c) Since $h(0)=5$, the constant term is 5. The next three terms are obtained by integrating the first three terms of the answer to part (b). The first four teams of the series for $h(x)$ are $5+x+\frac{x^3}{6}-\frac{x^5}{40}$.

28. (a) $a_0=1$

$$\begin{aligned}a_1&=\frac{3}{1}a_0=3\cdot 1=3\\a_2&=\frac{3}{2}a_1=\frac{3}{2}\cdot 3=\frac{9}{2}\\a_3&=\frac{3}{3}a_2=a_2=\frac{9}{2}\end{aligned}$$

28. Continued

(a) Since each term is obtained by multiplying the previous term by $\frac{3}{n}$, $a_n = \frac{3^n}{n!}$.

$$\sum_{n=0}^{\infty} a_n x^n = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{3}x^3 + \cdots + \frac{3^n}{n!}x^n + \cdots$$

(b) Since the series can be written as $\sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$, it represents the function $f(x) = e^{3x}$.

(c) $f'(1) = 3e^{3x}\Big|_{x=1} = 3e^3$

29. First, note that $\cos 18 \approx 0.6603$.

Using $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, enter the following two-step commands on your home screen and continue to hit ENTER.

```
0→N:1→T
                   1
N+1→N:T+(-1)^N*1
8^(2N)/(2N)!→T
                -161
                4213
            -43026.2
■
```

The sum corresponding to $N = 25$ is about 0.6582 (not within 0.001 of the exact value), and the sum corresponding to $N = 26$ is about 0.6606, which is within 0.001 of the exact value. Since we began with $N = 0$, it takes a total of 27 terms (or, up to and including the 52nd degree term).

30. One possible answer: Because the end behavior of a polynomial must be unbounded and $\sin x$ is not unbounded. Another: Because $\sin x$ has an infinite number of local extrema, but a polynomial can have only a finite number.

31. (1) $\sin x$ is odd and $\cos x$ is even
(2) $\sin 0 = 0$ and $\cos 0 = 1$

32. Replace x by $3x$ in series for $\sin x$. Therefore, we have $\frac{(3x)^5}{5!}$ so $\frac{3^5}{5!} = \frac{81}{40}$.

33. Since $\frac{d^3}{dx^3}\ln x = 2x^{-3}$, which is $\frac{1}{4}$ at $x = 2$, the coefficient is $\frac{\frac{1}{4}}{3!} = \frac{1}{24}$.

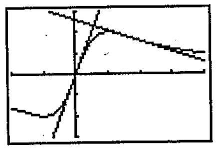
34. The linearization of f at a is the first order Taylor polynomial generated by f at $x = a$.

35. (a) Since $f'(x) = \frac{d}{dx}\frac{4x}{x^2+1}$

$$= \frac{(x^2+1)(4) - (4x)(2x)}{(x^2+1)^2}$$

$$= \frac{4 - 4x^2}{(x^2+1)^2},$$

we have $f(0) = 0$, $f'(0) = 4$, $f(\sqrt{3}) = \sqrt{3}$ and $f'(\sqrt{3}) = -\frac{1}{2}$, so the linearizations are $L_1(x) = 4x$ and $L_2(x) = \sqrt{3} - \frac{1}{2}(x - \sqrt{3}) = -\frac{1}{2}x + \frac{3}{2}\sqrt{3}$, respectively.



[–2, 4] by [–3, 3]

(b) $f''(a)$ must be 0 because of the inflection point, so the second degree term in the Taylor series of f at $x = a$ is zero.

36. The series represents $\tan^{-1} x$. When $x = 1$, it converges to $\tan^{-1} 1 = \frac{\pi}{4}$. When $x = -1$, it converges to $\tan^{-1}(-1) = -\frac{\pi}{4}$.

37. True. The constant term is $f(0)$.

38. False. It is –2 because the coefficient of x^3 is $\frac{f'''(0)}{3!}$.

39. E. $f(x) = 0 + x + 0 + \frac{2x^3}{3!}$

$$f(x) = \frac{1}{3}x^3 + x$$

40. A. $\frac{3^4}{4!} = \frac{27}{8}$

41. C.

42. A.

43. (a) $f(x) = \frac{1}{x}(\sin x)$

$$= \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots\right)$$

$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots + (-1)^n \frac{x^{2n}}{(2n+1)!} + \cdots$$

(b) Because f is undefined at $x = 0$.

(c) $k = 1$

44. Note that the Maclaurin series for $\frac{1}{1-x}$ is $1 + x + x^2 + \cdots + x^n + \cdots$. If we differentiate this series and multiply by x, we obtain the desired Maclaurin series $x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots$. Therefore, the desired function is $f(x) = x\frac{d}{dx}\frac{1}{1-x} = x\frac{1}{(1-x)^2} = \frac{x}{(x-1)^2}$.

45. (a) $f(x)=(1+x)^m$

$f'(x)=m(1+x)^{m-1}$

$f''(x)=m(m-1)(1+x)^{m-2}$

$f'''(x)=m(m-1)(m-2)(1+x)^{m-3}$

(b) Differentiating $f(x)$ k times gives

$f^{(k)}(x)=m(m-1)(m-2)\cdots(m-k+1)(1+x)^{m-k}$.

Substituting 0 or x, we have

$f^{(k)}(0)=m(m-1)(m-2)\cdots(m-k+1)$.

(c) The coefficient is

$$\frac{f^{(k)}(0)}{k!}=\frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}$$

(d) $f(0)=1, f'(0)=m,$ and we're done by part (c).

46. Because $f(x)=(1+x)^m$ is a polynomial of degree m.

Alternately, observe that $f^{(k)}(0)=0$ for $k\geq m+1$.

Section 9.3 Taylor's Theorem (pp. 495–502)

Exploration 1 Your Turn

1. We need to consider what happens to $R_n(x)$ as $n\to\infty$. By Taylor's Theorem, $R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-0)^n$, where $f^{(n+1)}(c)$ is the $(n+1)$st derivative of $\cos x$ evaluated at some c between x and 0. As with $\sin x$, we can say that $f^{(n+1)}(c)$ lies between –1 and 1 inclusive. Therefore, no matter what x is, we have

$$|R_n(x)|=\left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-0)^n\right|\leq\left|\frac{1}{(n+1)!}x^n\right|=\frac{|x|^n}{(n+1)!}.$$

The factorial growth in the denominator, as noted in Example 3, eventually outstrips the power growth in the numerator, and we have $\frac{|x|^n}{(n+1)!}\to 0$ for all x. This means that $R_n(x)\to 0$ for all x, which completes the proof.

Exploration 2 Euler's Formula

1. $$e^{ix}=1+ix+\frac{(ix)^2}{2!}+\frac{(ix)^3}{3!}+\cdots+\frac{(ix)^n}{n!}+\cdots$$
$$=1+ix-\frac{x^2}{2!}-i\frac{x^3}{3!}+\frac{x^4}{4!}+i\frac{x^5}{5!}-\cdots+(i)^n\frac{x^n}{n!}+\cdots$$

2. If we isolate the terms in the series that have i as a factor, we get:

$$e^{ix}=1+ix-\frac{x^2}{2!}-i\frac{x^3}{3!}+\frac{x^4}{4!}+i\frac{x^5}{5!}-\cdots+(i)^n\frac{x^n}{n!}+\cdots$$
$$=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots+i\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+(-1)^n\left(\frac{x^{2n+1}}{(2n+1)!}\right)+\cdots\right)$$
$$=\cos x+i\sin x.$$

(We are assuming here that we can rearrange the terms of a convergent series without affecting the sum. It happens to be true in this case, but we will see in Section 9.5 that it is not always true.)

3. $e^{i\pi}=\cos\pi+i\sin\pi=-1+0=-1$

Thus, $e^{i\pi}+1=0$

Quick Review 9.3

1. Since $|f(x)|=|2\cos(3x)|\leq 2$ on $[-2\pi, 2\pi]$ and $f(0)=2$, $M=2$.

2. Since $f(x)$ is increasing and positive on $[1, 2]$, $M=f(2)=7$.

3. Since $f(x)$ is increasing and positive on $[-3, 0]$, $M=f(0)=1$.

4. Since the minimum value of $f(x)$ is $f(-1)=-\frac{1}{2}$ and the maximum value of $f(x)$ is $f(1)=\frac{1}{2}$, $M=\frac{1}{2}$.

5. On $[-3, 1]$, the minimum value of $f(x)$ is $f(-3)=-7$ and the maximum value of $f(x)$ is $f(0)=2$. On $(1, 3]$, f is increasing and positive, so the maximum value of f is $f(3)=5$. Thus $|f(x)|\leq 7$ *on* $[-3,3]$ and $M=7$.

6. Yes, since each expression for an nth derivative given by the Quotient Rule will be a rational function whose denominator is a power of $x+1$.

7. No, since the function $f(x)=|x^2-4|$ has a corner at $x=2$.

8. Yes, since the derivatives of all orders for $\sin x$ and $\cos x$ are defined for all values of x.

9. Yes, since the function $f(x)=e^{-x}$ has derivatives of the form $f^{(n)}(x)=-e^{-x}$ for odd values of n and $f^{(n)}(x)=e^{-x}$ for even values of n, and both of these expressions are defined for all values of x.

10. No, since $f(x)=x^{3/2}$, we have $f'(x)=\frac{3}{2}x^{1/2}$ and $f''(x)=\frac{3}{4}x^{-1/2}$, so $f''(0)$ is undefined.