## **Section 9.3** Exercises

**1.** $f(0)=e^{-2x}\Big|_{x=0}=1$

$f'(0)=-2e^{-2x}\Big|_{x=0}=-2$

$f''(0)=4e^{-2x}\Big|_{x=0}=4$, so $\frac{f''(0)}{2!}=2$

$f'''(0)=-8e^{-2x}\Big|_{x=0}=-8$, so $\frac{f'''(0)}{3!}=-\frac{4}{3}$

$f^{(4)}(0)=16e^{-2x}\Big|_{x=0}=16$, so $\frac{f^{(4)}(0)}{4!}=\frac{2}{3}$

$P_4(x)=1-2x+2x^2-\frac{4}{3}x^3+\frac{2}{3}x^4$

$f(0.2)\approx P_4(0.2)=0.6704$

**2.** $f(0)=\cos\frac{\pi x}{2}\Big|_{x=0}=1$

$f'(0)=-\frac{\pi}{2}\sin\frac{\pi x}{2}\Big|_{x=0}=0$

$f''(0)=-\frac{\pi^2}{4}\cos\frac{\pi x}{2}\Big|_{x=0}=-\frac{\pi^2}{4}$, so $\frac{f''(0)}{2!}=-\frac{\pi^2}{8}$

$f'''(0)=\frac{\pi^3}{8}\sin\frac{\pi x}{2}\Big|_{x=0}=0$, so $\frac{f'''(0)}{3!}=0$

$f^{(4)}(0)=\frac{\pi^4}{16}\cos\frac{\pi x}{2}\Big|_{x=0}=\frac{\pi^4}{16}$, so $\frac{f^{(4)}(0)}{4!}=\frac{\pi^4}{384}$

$P_4(x)=1-\frac{\pi^2}{8}x^2+\frac{\pi^4}{384}x^4$

$f(0.2)\approx P_4(0.2)\approx 0.9511$

**3.** $f(0)=5\sin(-x)\big|_{x=0}=-5\sin x\big|_{x=0}=0$

$f'(0)=-5\cos x\big|_{x=0}=-5$

$f''(0)=5\sin x\big|_{x=0}=0$, so $\frac{f''(0)}{2!}=0$

$f'''(0)=5\cos x\big|_{x=0}=5$, so $\frac{f'''(0)}{3!}=\frac{5}{6}$

$f^{(4)}(0)=-5\sin x\big|_{x=0}=0$, so $\frac{f^{(4)}(0)}{4!}=0$

$P_4(x)=-5x+\frac{5}{6}x^3$

$f(0.2)\approx P_4(0.2)=-\frac{149}{150}\approx -0.9933$

**4.** Substituting $x^2$ for $x$ in the Maclaurin series given for $\ln(1+x)$ at the end of Section 9.2, we have

$$\ln(1+x^2)=x^2-\frac{(x^2)^2}{2}+\frac{(x^2)^3}{3}-\cdots+(-1)^{n-1}\frac{(x^2)^n}{n}+\cdots$$
$$=x^2-\frac{x^4}{2}+\frac{x^6}{3}-\cdots+(-1)^{n-1}\frac{x^{2n}}{n}+\cdots$$

Therefore, $P_4(x)=x^2-\frac{x^4}{2}$ and $f(0.2)\approx P_4(0.2)=0.0392$.

**5.** $f(0)=(1-x)^{-2}\Big|_{x=0}=1$

$f'(0)=2(1-x)^{-3}\Big|_{x=0}=2$

$f''(0)=6(1-x)^{-4}\Big|_{x=0}=6$, so $\frac{f''(0)}{2!}=3$

$f'''(0)=24(1-x)^{-5}\Big|_{x=0}=24$, so $\frac{f'''(0)}{3!}=4$

$f^{(4)}(0)=120(1-x)^{-6}\Big|_{x=0}=120$, so $\frac{f^{(4)}(0)}{4!}=5$

$P_4(x)=1+2x+3x^2+4x^3+5x^4$

$f(0.2)\approx P_4(0.2)=1.56$

**6.** $\sin x-x+\frac{x^3}{3!}$

$$=\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots\right)-x+\frac{x^3}{3!}$$
$$=\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots$$

Note: By replacing $n$ with $n+2$, the general term can be written as $(-1)^n\frac{x^{2n+5}}{(2n+5)!}$

**7.** $$xe^x=x\left(1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots\right)$$
$$=x+x^2+\frac{x^3}{2!}+\cdots+\frac{x^{n+1}}{n!}+\cdots$$

**8.** $\cos^2 x=\frac{1}{2}+\frac{1}{2}\cos(2x)$

$$=\frac{1}{2}+\frac{1}{2}\left(1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\cdots+(-1)^n\frac{(2x)^{2n}}{(2n)!}+\cdots\right)$$
$$=1-\frac{4x^2}{2\cdot 2!}+\frac{16x^4}{2\cdot 4!}-\cdots+(-1)^n\frac{2^{2n}x^{2n}}{2\cdot(2n)!}+\cdots$$
$$=1-x^2+\frac{x^4}{3}-\cdots+(-1)^n\frac{2^{2n-1}x^{2n}}{(2n)!}+\cdots$$

**9.** $\sin^2 x=\frac{1}{2}-\frac{1}{2}\cos(2x)$

$$=\frac{1}{2}-\frac{1}{2}\left(1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}+\cdots+(-1)^n\frac{(2x)^{2n}}{(2n)!}+\cdots\right)$$
$$=\frac{4x^2}{2\cdot 2!}-\frac{16x^4}{2\cdot 4!}+\frac{64x^6}{2\cdot 6!}-\cdots+(-1)^{n-1}\frac{2^{2n}x^{2n}}{2\cdot(2n)!}+\cdots$$
$$=x^2-\frac{x^4}{3}+\frac{2x^6}{45}-\cdots+(-1)^{n-1}\frac{2^{2n-1}x^{2n}}{(2n)!}+\cdots$$

Note: By replacing $n$ with $n+1$, the general term can be written as $(-1)^n\frac{2^{2n+1}x^{2n+2}}{(2n+2)!}$.

10. $\dfrac{x^2}{1-2x}=x^2\left(\dfrac{1}{1-2x}\right)$

$=x^2[1+2x+(2x)^2+\cdots+(2x)^n+\cdots]$
$=x^2+2x^3+4x^4+\cdots+2^n x^{n+2}+\cdots$

11. $P_7(x)$

12. $P_{12}(x)$

13. $(2|x|)^7$

14. $\dfrac{x^{10}}{1-x}$

15. $|f^{(n+1)}(t)|=te^t\le Mr^{n+1}$

$M=te^t$
$[x,0]$
$M=0$
$[0,x]$
$M=xe^x$
$te^t\le M$

16. $|f^{(n+1)}(t)|=\sin t-t+\dfrac{t^3}{3!}\le Mr^{n+1}$

$M=\sin t-t+\dfrac{t^3}{3!}$
$[x,0]$
$M=0$
$[0,x]$
$M=\sin x-x+\dfrac{x^3}{3!}$
$\sin t-t+\dfrac{t^3}{3!}\le M$

17. $|f^{(n+1)}(t)|=\sin^2 t\le Mr^{n+1}$

$M=\sin^2 t$
$[x,0]$
$M=0$
$[0,x]$
$M=\sin x$
$\sin^2 t\le M$

18. $|f^{(n+1)}(t)|=\cos^2 t\le Mr^{n+1}$

$M=\cos^2 t$
$[x,0]$
$M=1$
$[0,x]$
$M=\cos^2 x$
$\cos^2 t\le M$

19. Let $f(x)=\sin x$. Then $P_4(x)=P_3(x)=x-\dfrac{x^3}{6}$, so we use the Remainder Estimation Theorem with $n=4$. Since $|f^{(5)}(x)|=|\cos x|\le 1$ for all $x$, we may use $M=r=1$, giving $|R_4(x)|\le\dfrac{|x|^5}{5!}$, so we may assure that $|R_4(x)|\le 5\times 10^{-4}$ by requiring $\dfrac{|x|^5}{5!}\le 5\times 10^{-4}$, or $|x|\le\sqrt[5]{0.06}\approx 0.5697$. Thus, the absolute error is no greater than $5\times 10^{-4}$ when $-0.56<x<0.56$ (approximately).

Alternate method: Using graphing techniques,

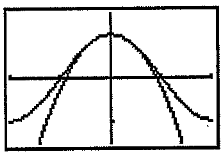
$$\left|\sin x-\left(x-\frac{x^3}{6}\right)\right|\le 5\times 10^{-4}\text{ when } -0.57<x<0.57.$$

20. Let $f(x)=\cos x$. Then $P_3(x)=P_2(x)=1-\dfrac{x^2}{2}$, so we may use the Remainder Estimation Theorem with $n=3$. Since $|f^{(4)}(x)|=|\cos x|\le 1$ for all $x$, we may use $M=r=1$, giving $|R_3(x)|\le\dfrac{|x|^4}{4!}$. For $|x|<0.5$, the absolute error is less than $\dfrac{(0.5)^4}{4!}\approx 0.0026$ (approximately).

Alternate method: Using graphing techniques, we find that when $|x|<0.5$,

$$|\text{error}|=\left|\cos x-\left(1-\frac{x^2}{2}\right)\right|$$
$$<\left|\cos 0.5-\left(1-\frac{0.5^2}{2}\right)\right|$$
$$\approx 0.002583.$$

The quantity $1-\dfrac{x^2}{2}$ tends to be too small, as shown by the graphs of $y=\cos x$ and $y=1-\dfrac{x^2}{2}$.



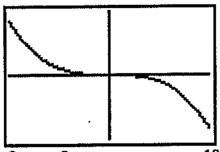
$[-\pi,\pi]$ by $[-1.5, 1.5]$

21. Let $f(x)=\sin x$. Then $P_2(x)=P_1(x)=x$, so we may use the Remainder Estimation Theorem with $n=2$. Since $|f'''(x)|=|-\cos x|\le 1$ for all $x$, we may use $M=r=1$, giving $|R_2(x)|\le\dfrac{|x|^3}{3!}$. Thus, for $|x|<10^{-3}$, the maximum possible error is about $\dfrac{(10^{-3})^3}{3!}\approx 1.67\times 10^{-10}$.

Alternate method:
Using graphing techniques, we find that when $|x|<10^{-3}$, $|\text{error}|=|\sin x-x|\le|\sin 10^{-3}-10^{-3}|\approx 1.67\times 10^{-10}$.

**21. Continued**

The inequality $x < \sin x$ is true for $x < 0$, as we may see by graphing $y = \sin x - x$.



$[-10^{-3}, 10^{-3}]$ by $[-2 \times 10^{-10}, 2 \times 10^{-10}]$

**22.** Let $f(x) = \sqrt{1+x}$. Then $P_1(x) = 1 + \frac{x}{2}$, so we may use the Remainder Estimation Theorem with $n = 1$. Since $|f''(x)| = \left|-\frac{1}{4}(1+x)^{-3/2}\right|$, which is less than 0.2538 for $|x| < 0.01$, we may use $M = 0.2538$ and $r = 1$, giving $|R_1(x)| \le \frac{0.2538|x|^2}{2!}$. Thus, for $|x| < 0.01$ the maximum Possible absolute error is about $\frac{0.2538(0.01)^2}{2!} \approx 1.27 \times 10^{-5}$.

Alternate method:

Using graphing techniques, we find that when $|x| < 0.01$,

$$|\text{error}| = \left|\sqrt{1+x} - \left(1 + \frac{x}{2}\right)\right|$$
$$\le \left|\sqrt{1-0.01} - \left(1 - \frac{0.01}{2}\right)\right|$$
$$\approx 1.26 \times 10^{-5}.$$

**23.** Note that $1 + x + \frac{x^2}{2}$ is the second order Taylor polynomial for $f(x) = e^x$ at $x = 0$, so we may use the Remainder Estimation Theorem with $n = 2$. Since $|f'''(x)| = e^x$, which is less than $e^{0.1}$ when $|x| < 0.1$ and $r = 1$, giving $|R_2(x)| \le \frac{e^{0.1}|x|^3}{3!}$. Thus, for $|x| < 0.1$, the maximum possible error is about $\frac{e^{0.1}(0.1)^3}{3!} \approx 1.842 \times 10^{-4}$.

**24.** Note that $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$ and $e^{-x} = 1 - x + \frac{x^2}{2!} - \cdots + (-1)^n \frac{x^n}{n!} + \cdots$ Thus the terms with $n$ even will cancel for $\sinh x = \frac{1}{2}(e^x - e^{-x})$, and the terms with $n$ odd will cancel for $\cosh\ x = \frac{1}{2}(e^x - e^{-x})$.

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + \ldots$$
$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!} + \ldots$$

**25.** All of the derivatives of $\cosh x$ are either $\cosh x$ or $\sinh x$. For any real $x$, $\cosh x$ and $\sinh x$ are both bounded by $e^{|x|}$. So for any real $x$, let $M = e^{|x|}$ and $r = 1$ in the Remainder Estimation Theorem. This gives $|R_n(x)| \le \frac{e^{|x|} x^{n+1}}{(n+1)!}$ But for any fixed value of $x$, $\lim_{n\to\infty} \frac{e^{|x|}|x|^{n+1}}{(n+1)!} = 0$. It follows that the series converges to $\cosh x$ for all real values of $x$.

**26.** For $n = 0$, Taylor's Theorem with Remainder says that if $f$ has derivatives of all orders in an open interval $I$ containing $a$, then for each $x$ in $I$, $f(x) = f(a) + R(x)$, where $R(x) = f'(c)(x-a)$, so $f(x) = f(a) + f'(c)(x-a)w$ for some $c$ between $a$ and $x$. Letting $b = x$ this equation is $f(b) = f(a) + f'(c)(b-a)$, which is equivalent to $f'(c) = \frac{f(b)-f(a)}{b-a}$ for some $c$ between $a$ and $b$. Thus, for the class of functions that have derivatives of all orders in an open interval containing $a$ and $b$, the Mean Value Theorem can be considered a special case of Taylor's Theorem.

**27.** $f(0) = \ln(\cos x)\big|_{x=0} = \ln\ 1 = 0$

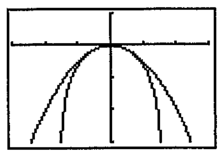
$f'(0) = \frac{1}{\cos x}(-\sin x)\big|_{x=0} = -\tan x\big|_{x=0} = 0$

$f''(0) = -\sec^2 x\big|_{x=0} = -1$ so $\frac{f''(0)}{2!} = -\frac{1}{2}$

**(a)** $L(x) = 0$

**(b)** $P_2(x) = -\frac{1}{2}x^2$

**(c)** The graphs of the linear and quadratic approximations fit the graph of the function near $x = 0$.



$[-3, 3]$ by $[-3, 1]$

**28.** $f(0) = e^{\sin x}\big|_{x=0} = e^0 = 1$

$f'(0) = e^{\sin x}\cos x\big|_{x=0} = 1$

$f''(0) = \left[(e^{\sin x})(-\sin x) + (\cos x)(e^{\sin x}\cos x\right]\big|_{x=0} = 1,$

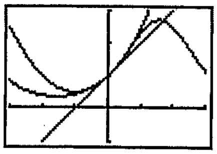
so $\frac{f''(0)}{2!} = \frac{1}{2}$

**(a)** $L(x) = 1 + x$

**(b)** $P_2(x) = 1 + x + \frac{x^2}{2}$

**28. Continued**

(c) The graphs of the linear and quadratic approximations fit the graph of the function near $x = 0$.



[–3, 3] by [–1,3]

**29.** $f(0) = (1-x^2)^{-1/2}\big|_{x=0} = 1$

$f'(0) = -\frac{1}{2}(1-x^2)^{-3/2}(-2x)\big|_{x=0} = x(1-x^2)^{-3/2}\big|_{x=0} = 0$

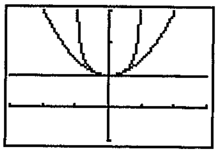
$f''(0) = (x)\left[-\frac{3}{2}(1-x^2)^{-5/2}(-2x)\right] + (1-x^2)^{-3/2}\big|_{x=0} = 1,$

so $\frac{f''(0)}{2!} = \frac{1}{2}$

(a) $L(x) = 1$

(b) $P_2(x) = 1 + \frac{x^2}{2}$

(c) The graphs of the linear and quadratic approximations fit the graph of the function near $x = 0$.



[–3, 3] by [–1, 3]

**30.** $f(0) = \sec x\big|_{x=0} = 1$

$f'(0) = \sec x \tan x\big|_{x=0} = 0$

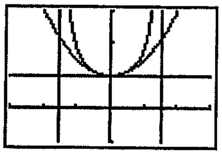
$f''(0) = (\sec x)(\sec^2 x) + (\tan x)(\sec x \tan x)\big|_{x=0} = 1,$

so $\frac{f''(0)}{2!} = \frac{1}{2}$

(a) $L(x) = 1$

(b) $P_2(x) = 1 + \frac{x^2}{2}$

(c) The graphs of the linear and quadratic approximations fit the graph of the function near $x = 0$.



[–3, 3] by [–1, 3]

**31.** $f(0) = \tan x\big|_{x=0} = 0$

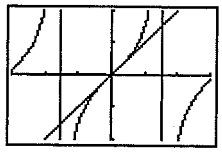
$f'(0) = \sec^2 x\big|_{x=0} = 1$

$f''(0) = (2\sec x)(\sec x \tan x)\Big|_{x=0} = 0$, so $\frac{f''(0)}{2!} = 0$

(a) $L(x) = x$

(b) $P_2(x) = x$

(c) The graphs of the linear and quadratic approximations fit the graph of the function near $x = 0$.



[–3, 3] by [–2, 2]

**32.** $f(0) = (1+x)^k\big|_{x=0} = 1$

$f'(0) = k(1+x)^{k-1}\big|_{x=0} = k$

$f''(0) = k(k-1)(1+x)^{k-2}\big|_{x=0} = k(k-1),$

so $\frac{f''(0)}{2!} = \frac{k(k-1)}{2}$

$P_2(x) = 1 + kx + \frac{k(k-1)}{2}x^2$

For $k = 3$, we have $f(x) = (1+x)^3$ and $f'''(x) = 6$. We may use the Remainder Estimation Theorem with $n = 2$, $M = 6$, and $r = 1$, giving $R_2(x) \le \frac{6|x|^3}{3!} = |x|^3$. (In this particular case it is actually true that $R_2(x) = x^3$, since $f(x)$ is a cubic polynomial.) Thus the absolute error is less than $\frac{1}{100}$ whenever $|x|^3 < 0.01$. In the interval [0, 1], this occurs when $0 \le x < \sqrt[3]{0.01} \approx 0.215$.

Alternate method:

Note that $P_2(x) = 1 + 3x + 3x^2$. Using graphing techniques, $|(1+x)^3 - (1+3x+3x^2)| < \frac{1}{100}$ when $|x| < 0.215$.

**33.** Let $f(x) = e^x$. Then $P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$, so we may use the Remainder Estimation Theorem with $n = 3$. Since $|f^{(4)}(x)| = e^x$, which is no more than $e^{0.1}$ when $|x| \le 0.1$, we may use $M = e^{0.1}$ and $r = 1$, giving $|R_3(x)| \le \frac{e^{0.1}|x|^4}{4!}$. Thus, for $|x| \le 0.1$, the maximum possible absolute error is about $\frac{e^{0.1}(0.1)^4}{24} \approx 4.605 \times 10^{-6}$.

Alternate method:

Using graphing techniques, when $|x| \le 0.1$,

$$|\text{error}| = \left|e^x - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\right|$$

$$\le \left|e^{0.1} - \left(1 + 0.1 + \frac{0.01}{2} + \frac{0.001}{6}\right)\right|$$

$$\approx 4.251 \times 10^{-6}.$$

**34.** Since the Maclaurin series is

$$\frac{1}{1-x}=1+x+x^2+\cdots+x^n+\cdots,$$

$$P_3(x)=1+x+x^2+x^3.$$

Since $\left|f^{(4)}(x)\right|=24(1-x)^{-5}$, which is no more than $24(0.9)^{-5}$ when $|x|\le 0.1$, we may use $M=24(0.9)^{-5}$ and $r=1$, giving $|R_3(x)|\le\frac{24(0.9)^{-5}|x|^4}{4!}=\frac{|x|^4}{0.9^5}$. Thus, for $|x|\le 0.1$, an upper bound for the magnitude of the approximation error is $\frac{0.1^4}{0.9^5}\approx 1.694\times 10^{-4}$. Rounding up to be safe, an upper bound is $1.70\times 10^{-4}$.

Alternate method:

Using graphing techniques, when $|x|\le 0.1$,

$$|\text{error}|=\left|\frac{1}{1-x}-(1+x+x^2+x^3)\right|$$

$$\le\left|\frac{1}{1-0.1}-1.111\right|\approx 1.11\times 10^{-4}.$$

**35.** **(a)** No

**(b)** Yes, since

$$\frac{dy}{dx}=e^{-x^2}$$

$$=1+(-x^2)+\frac{(-x^2)^2}{2!}+\cdots+\frac{(-x^2)^n}{n!}+\cdots$$

$$=1-x^2+\frac{x^4}{2!}-\cdots+(-1)^n\frac{x^{2n}}{n!}+\cdots.$$

The constant term of $y$ is $y(0)=2$, and we may obtain the remaining terms of $y$ by integrating the above series.

$$y=2+x-\frac{x^3}{3}+\frac{x^5}{10}-\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)n!}+\cdots$$

By substituting $n-1$ for $n$, the general term may also be written as $(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)(n-1)!}$.

**(c)** The power series equals the function $y$ for all real values of $x$. This is because the series for $e^{-x^2}$ converges for all real values of $x$, so Theorem 2 of Section 9.1 implies that the new series also converges for all $x$.

**36.** **(a)** Substitute $-x$ for $x$ in the Maclaurin series for $\ln(1+x)$ given at the end of Section 9.2.

$$\ln(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots-\frac{x^n}{n}-\cdots$$

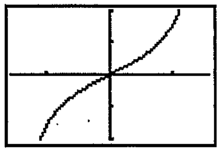
**(b)**

$$\ln\frac{1+x}{1-x}=\ln(1+x)-\ln(1-x)$$

$$=\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+(-1)^{n-1}\frac{x^n}{n}+\cdots\right)+\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{x^n}{n}\right)$$

$$=2x+\frac{2x^3}{3}+\frac{2x^5}{5}+\cdots+\frac{2x^{2n+1}}{2n+1}+\cdots$$

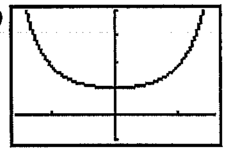
**37.** **(a)**



$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ by $[-2, 2]$

The series approximates $\tan x$.

**(b)**



$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ by $[-1, 4]$

The series approximates $\sec x$.

**38.** False. If $f'(a)$ happens to be 0, the linearization is a constant function.

**39.** True. The coefficient of $x$ is $f'(0)$.

**40.** D. $1.5-\frac{1.5^3}{3!}+\frac{1.5^5}{5!}=1.001$

**41.** E.

**42.** B.

**43.** A.

**44.** **(a)**

$$\sin^2 x=\frac{1}{2}(1-\cos 2x)$$

$$=\frac{1}{2}-\frac{1}{2}\left(1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}+\cdots+(-1)^n\frac{(2x)^{2n}}{(2n)!}+\cdots\right)$$

$$=\frac{4x^2}{2\cdot 2!}-\frac{16x^4}{2\cdot 4!}+\frac{64x^6}{2\cdot 6!}-\frac{256x^8}{2\cdot 8!}+\frac{1024x^{10}}{2\cdot 10!}-\cdots$$

$$=x^2-\frac{x^4}{3}+\frac{2x^6}{45}-\frac{x^8}{315}+\frac{2x^{10}}{14{,}175}-\cdots$$

**(b)** derivative $=2x-\frac{4x^3}{3}+\frac{4x^5}{15}-\frac{8x^7}{315}+\cdots$

**(c)** part (b) $=2x-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}-\frac{(2x)^7}{7!}+\cdots=\sin 2x$

**45.** **(a)** It works. For example, let $n = 2$. Then $P = 3.14$ and $P + \sin P \approx 3.141592653$, which is accurate to more than 6 decimal places.

**(b)** Let $P = \pi + x$ where $x$ is the error in the original estimate. Then

$P + \sin P = (\pi + x) + \sin(\pi + x) = \pi + x - \sin x$

But by the Remainder Theorem, $|x - \sin x| < \frac{|x|^3}{6}$.

Therefore, the difference between the new estimate $P + \sin P$ and $\pi$ is less than $\frac{|x|^3}{6}$.

**46.** **(a)**
$$\frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{(\cos\theta + i\sin\theta) + (\cos(-\theta) + i\sin(-\theta))}{2} = \frac{\cos\theta + i\sin\theta + \cos\theta - i\sin\theta}{2} = \frac{2\cos\theta}{2} = \cos\theta$$

**(b)**
$$\frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{(\cos\theta + i\sin\theta) - (\cos(-\theta) + i\sin(-\theta))}{2i} = \frac{(\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)}{2i} = \frac{2i\sin\theta}{2i} = \sin\theta$$

**47.**
$$\frac{d}{dx}\left[e^{ax}(\cos bx + i\sin bx)\right]$$
$$= (e^{ax})(-b\sin bx + bi\cos bx) + (ae^{ax})(\cos bx + i\sin bx)$$
$$= (e^{ax})\left[(bi^2\sin bx + bi\cos bx) + a(\cos bx + i\sin bx)\right]$$
$$= (e^{ax})\left[bi(\cos bx + i\sin bx) + a(\cos bx + i\sin bx)\right]$$
$$= (a + bi)(e^{ax})(\cos bx + i\sin bx)$$
$$= (a + bi)e^{(a+bi)x}$$

**48.** **(a)** The derivative of the right-hand side is
$$\frac{a-bi}{a^2+b^2}(a+bi)e^{(a+bi)x} = \frac{a^2-(bi)^2}{a^2+b^2}e^{(a+bi)x} = \frac{a^2+b^2}{a^2+b^2}e^{(a+bi)x} = e^{(a+bi)x},$$
which confirms the antiderivative formula.

**(b)**
$$\int e^{ax}\cos bx\,dx + i\int e^{ax}\sin bx\,dx$$
$$= \int e^{(a+bi)x}\,dx$$
$$= \frac{a-bi}{a^2+b^2}e^{(a+bi)x}$$
$$= \frac{a-bi}{a^2+b^2}e^{ax}(\cos bx + i\sin bx)$$
$$= \left(\frac{e^{ax}}{a^2+b^2}\right)(a\cos bx + b\sin bx - bi\cos bx + ai\sin bx)$$
$$= \left(\frac{e^{ax}}{a^2+b^2}\right)\left[(a\cos bx + b\sin bx) + i(a\sin bx - b\cos bx)\right]$$

Separating the real and imaginary parts gives
$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) \text{ and}$$
$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

## Quick Quiz Sections 9.1–9.3

**1.** D.

**2.** A. $2 - 1x + \frac{6x^2}{3!} + \frac{12x^3}{4!}$
$= 2 - x + 3x^2 + 2x^3$

**3.** E.

**4.** **(a)** Since the series is geometric, it converges if and only if $|r| < 1$, where $r = \frac{x+2}{3}$.

$\left|\frac{x+2}{3}\right| < 1 \Rightarrow |x+2| < 3 \Rightarrow -5 < x < 1$. The interval of convergence is (–5, 1).

**(b)** The series is geometric with first term 2 and common ratio $r = \frac{x+2}{3}$. It therefore converges to
$$\frac{2}{1 - \frac{x+2}{3}} = \frac{6}{1-x}.$$

## Section 9.4 Radius of Convergence (pp. 503–512)

### Exploration 1 Finishing the Proof of the Ratio Test

**1.** For $\sum \frac{1}{n}$: $L = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$

For $\sum \frac{1}{n^2}$: $L = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1.$